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STRUCTURAL RELIABILITY UNDER CONDITIONS OF FATIGUE AND ULTIMATE LOAD FAILURE

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Ohio State University Research Foundation

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AIR FORCE MATERIALS LABORATORY
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WRIGHT-PATTERSON AIR FORCE BASE, OHIO

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FOREWORD

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ABSTRACT

On the basis of assumed additivity of risk functions, the statistical distribution of the time to first failure of a group of nominally identical structures, each with two critical locations respectively subjected to chance failure and fatigue failure, is derived together with analytical expressions of other statistical quantities pertinent to the problem of the time to first failure. The concept of fatigue sensitivity is examined. An example involving aircraft wings designed according to the current method of design is worked out to demonstrate how the reliability of such wings can be estimated. The example shows a dominance of chance failure if the group size increases and/or a higher reliability is demanded. It is pointed out that the present method of reliability estimation can easily be extended to cover more realistic structures.

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LIST OF SYMBOLS

C	=	chance failure
ET	=	$ET^{(1)}$; expected life of structure
ET_C	=	$ET_C^{(1)}$; expected life of reference structure free from fatigue failure
ET_F	=	$ET_F^{(1)}$; expected life of reference structure free from chance failure
$E_C T$	=	$E_C T^{(1)}$; (conditional) expected life of structure failing under mode C
$E_F T$	=	$E_F T^{(1)}$; (conditional) expected life of structure failing under mode F
$ET^{(n)}$	=	expected time to first failure among a group of n structures
$ET_C^{(n)}$	=	expected time to first failure among a group of n reference structures free from fatigue failure
$ET_F^{(n)}$	=	expected time to first failure among a group of n reference structures free from chance failure
$E_C T^{(n)}$	=	(conditional) expected time to first failure of the group with first failure under mode C
$E_F T^{(n)}$	=	(conditional) expected time to first failure of the group with first failure under mode F
EN, EN_i	=	expected life in constant load fatigue in terms of number of load cycles
EN_R	=	expected life in random fatigue in terms of number of load cycles
F	=	fatigue failure
F(u)	=	distribution function of gust velocity u
f(t)dt	=	$f^{(1)}(t)dt$; probability that structure will fail in [t,t+dt)
$f_C(t)dt$	=	$f_C^{(1)}(t)dt$; probability that structure will fail in [t,t+dt) under mode C

$f_F(t)$	=	$f_F^{(1)}(t)dt$; probability that structure will fail in [t,t+dt) under mode F
$f^{(n)}(t)dt$	=	probability that first failure among a group of n structures will occur [t,t+dt)
$f_C^{(n)}(t)dt$	=	probability that first failure among a group of n structures will occur in [t,t+dt) under mode C
$f_F^{(n)}(t)dt$	=	probability that first failure among a group of n structures will occur in [t,t+dt) under mode F
$h(t)dt$	=	$h^{(1)}(t)dt$; probability that structure which has survived up to t, will fail in [t,t+dt)
$h_C(t)dt$	=	$h_C^{(1)}(t)$; probability that structure which has survived up to t, will fail in [t,t+dt) under mode C
$h_F(t)dt$	=	$h_F^{(1)}(t)dt$; probability that structure which has survived up to t will fail in [t, t+dt) under mode F
$h^{(n)}(t)dt$	=	probability that first failure will occur in [t,t+dt) assuming that no failure has occurred to a group of n structures up to t.
$h_C^{(n)}(t)dt$	=	probability that first failure will occur in [t,t+dt) under mode C assuming that no failure has occurred to a group of n structures up to t
$h_F^{(n)}(t)dt$	=	probability that first failure will occur in [t,t+dt) under mode F assuming that no failure has occurred to a group of n structures up to t
K_n	=	gust sensitivity factor, Eq. (60)
$k_C, k_C^{(n)}$	=	fatigue sensitivity, Eqs. (45) and (47)
$k_F^{(n)}$	=	chance-failure sensitivity, Eq. (49)
$L(t)$	=	$L^{(1)}(t)$; probability that structure will survive up to t
$L_C(t)$	=	$L_C^{(1)}(t)$; probability that fatigue-failure-free reference structure will survive up to t
$L_F(t)$	=	$L_F^{(1)}(t)$; probability that chance-failure-free reference structure will survive up to t
$L^{(n)}(t)$	=	probability that first failure among a group of n structures will not occur before t

$L_C^{(n)}(t)$ = probability that first failure among a group of n fatigue-failure-free reference structures, will not occur before t

$L_F^{(n)}(t)$ = probability that first failure among a group of n chance-failure-free reference structures, will not occur before t

$L_F{}^{*(n)}(t)$ = conditional probability that time to first failure of the group of n structures with first failure under mode F, is larger than t

m = number of stress levels

N_e = characteristic life in number of stress cycles, Eq. (64)

N_o = number of stress cycles before which no fatigue damage is accumulated, Eq. (64)

r = group size or number of structures in group

O_C = location of chance failure

O_F = location of fatigue failure

P_C = $P_C^{(1)}$; probability that structure will fail under mode C

P_F = $P_F^{(1)}$; probability that structure will fail under mode F

$P_C^{(n)}$ = probability that first failure among a group of n structures will occur under mode C

$P_F^{(n)}$ = probability that first failure among a group of n structures will occur under mode F

T = $T^{(1)}$; random variable representing life of structure

$T^{(n)}$ = random variable representing time to first failure among a group of n structures

T_C = $T_C^{(1)}$; random variable representing life of fatigue-failure-free reference structure

T_F = $T_F^{(1)}$; random variable representing life of chance-failure-free reference structure

$T_C^{(n)}$	=	random variable representing time to first failure among a group of n fatigue-failure-free reference structures
$T_F^{(n)}$	=	random variable representing time to first failure among a group of n chance-failure-free reference structures
UFL	=	(actual) ultimate failure load
UFLN	=	nominal ultimate failure load
u	=	gust velocity (ft./sec.)
u_D	=	minimum gust velocity to produce ultimate failure
V	=	flight speed of aircraft
V_C	=	expected life of structure subjected to chance failure only (expected life of fatigue-failure-free reference structure)
V_F, V_F^*	=	expected life of structure subjected to fatigue failure only (expected life of chance-failure-free reference structure)
v	=	coefficient of variation of $T^{(n)}$
α, α^*	=	scale parameter in the Weibull distribution, Eq. (64)
$\alpha-1, \alpha^*-1$	=	power of rate of fatigue failure, Eq. (2)
γ	=	V_F/V_C
λ	=	$\gamma n^{1-1/\alpha}$
$\nu_o(u)$	=	frequency of gust cycles with velocity larger than u per mile
$\nu(u)$	=	frequency of gust cycles with velocity larger than u per hour
$\nu^*(u)$	=	$\nu(u)/\nu(0)$; relative frequency of gust cycles with velocity larger than u

SECTION I

INTRODUCTION

Significance of the time to first failure (minimum life among a group of nominally identical structures each subjected to a sample sequence of the random load) as one of most important design parameters from the view point of structural reliability is emphasized by A. M. Freudenthal.[1] The emphasis is essentially based on the following grounds: (1) The implication of a high percentage of number of structural failures before the mean time to failure precludes the use of the mean time to failure as a design parameter, (2) the time to first failure is of major concern in the engineering as well as economical and other decision making, (3) mathematically it has a statistical distribution of extremes and hence the use of its asymptotic form is justified if a large number of structures are involved, and (4) its reliability demonstration is not completely out of question even with major structures such as an aircraft.

If the structure to be considered has only one critical location with a mode of failure identified by a power law of failure rate (chance failure is a particular case of this), or with a mode of failure associated with the log-normal or the gamma life distribution, then the statistical distribution of the time to first failure among a group of these structures can be computed without much difficulty.[2]

It is, however, evident that such a simple model is not valid for a structure such as an aircraft wing where a large number of locations structurally sensitive to fatigue damage as well as chance failure have to be considered. Furthermore, the effect of interaction between such chance and fatigue failure on the time to first failure is not clearly known at the present time.

In the following section (Section II), therefore, a clear analytical formulation of the problem is given and the effect of interaction between these two modes of failure on the time to first failure is iden-

tified. In doing so, a structural model is considered that consists of two critical failure components, one in chance failure and one in fatigue. This is an idealization of the situation in which the root of an aircraft wing has a possibility of failure in instability on the upper surface whereas it is sensitive to fatigue failure on the lower surface.

In Section III, the definition of fatigue sensitivity, a notion originated by A. M. Freudenthal[3] is reexamined and the chance-failure sensitivity is also introduced.

Taking the aircraft wing mentioned above as an example, it is demonstrated in Section IV, (1) how the distribution of the time to first failure among a group of n such wings designed according to the current method of design, can be estimated, (2) what effect on the reliability is to be expected if the group size is increased and (3) what reliability statements can be made in general.

The problem of how to deal with more realistic cases involving a number of critical locations is discussed, among other items, in Section V, suggesting that the method developed here can be applied to the reliability estimate of real wing structures if these critical locations and their modes of failure as well as the failure rates are identified and assessed by expert engineers familiar with structural details on the basis of available analytical and experimental knowledge combined with their experience.

It is emphasized that the reliability estimate of a structure cannot be better than the present state of art of structural analysis, prediction of its operating conditions and estimation of mechanical properties of the material used. The fact that a number of explicit and quantitative assumptions have to be made in order to arrive at a final reliability figure, is not to be considered as a weakness inherent to the reliability analysis. Rather, it is to be considered as a clear and conscientious recognition of the problem areas in which

further study, analytical and experimental, and statistical data collection are needed. In fact, it is only through the reliability analysis that the importance of these assumptions or the significance of additional information can be assessed in direct relation to the structural safety.

SECTION II

BASIC PROBABILISTIC ANALYSIS

Let C (F) denote a chance failure (a fatigue failure) at location O_C (O_F) within the structure. Furthermore, let $h_C(t)$ $\left(h_F(t)\right)$ denote the risk function or the rate of failure for the failure mode C (the failure mode F). By definition, $h_C(t)\left(h_F(t)\right)$ is such that $h_C(t)dt(h_F(t)dt)$ is the conditional probability of failure under C(F) in the time interval [t,t+dt) given that $O_C(O_F)$ has survived the interval [0,t).

The chance failure implies a constant risk and hence

$$h_C(t) = \frac{1}{V_C} \tag{1}$$

whereas it is assumed as usual on the basis of available data that the fatigue failure is a process with the following risk function with $\alpha = 2 \sim 5$.

$$h_F(t) = \frac{\alpha}{V_F}\left(\frac{t}{V_F}\right)^{\alpha-1} \tag{2}$$

where V_C and V_F are the characteristic lives of O_C and O_F respectively and the scale parameter α is an inverse measure of dispersion of the fatigue life distribution.

It follows from the definition of the risk function that

$$h_F(t) = -\frac{d\,\ell n\,L_F(t)}{dt} = -\frac{d\,L_F(t)}{dt} \Big/ L_F(t) \tag{3}$$

$$h_C(t) = -\frac{d\,\ell n\,L_C(t)}{dt} = -\frac{d\,L_C(t)}{dt} \Big/ L_C(t) \tag{4}$$

or

$$L_F(t) = \exp\left[-\int_0^t h_F(u)du\right] = \exp\left[-\left(\frac{t}{V_F}\right)^{\alpha}\right] \tag{5}$$

= probability that O_F will survive the interval [0,t), or, more rigorously, probability that a chance-failure-free reference structure will survive [0,t).

$$L_C(t) = \exp[-\int_0^t h_C(u)du] = \exp[-(\frac{t}{V_C})] \tag{6}$$

= probability that O_C will survive the interval $[0,t)$ (survival of a fatigue-failure-free reference structure)

where the initial condition $L_F(0) = L_C(0) = 1$ is assumed.

Since the survival of the structure requires the survivals of both O_C and O_F,

$$\begin{aligned} L(t) = L_F(t).L_C(t) &= \exp[-\int_0^\infty \{h_F(t)+h_C(t)\}dt] \\ &= \exp[-(\frac{t}{V_C}) - (\frac{t}{V_F})^\alpha] \end{aligned} \tag{7}$$

= probability of survival of the structure (as a whole) in $[0,t)$

From Eq. (7), it follows that the failure rate $h(t)$ of the structure is

$$h(t) = h_C(t) + h_F(t) = \frac{1}{V_C} + \frac{\alpha}{V_F}(\frac{t}{V_F})^{\alpha-1} \tag{8}$$

This indicates that the probability of failure of the structure in the time interval $[t,t+dt)$ (due to the failure mode C and/or the mode F) is the sum of the probabilities of failure at O_C and O_F in the same time interval. Therefore, it implies that the failures at O_C and O_F are mutually exclusive. It is pointed out that such mutual exclusiveness follows from the independence assumption of the failure processes at O_C and O_F tacitly employed in deriving Eq. (7).

The assumption of independence and hence the assumption of the mutual exclusiveness is not entirely correct since O_C and O_F are the parts of the same structure and therefore are subjected to the same sequence of load. Nevertheless, Eq. (8) is assumed in the present study because (1) it is an upper bound of the failure rate of the structure (this can be shown by a direct use of the addition rule of

probability) and (2) it simplifies the analysis significantly. The following analysis provides a conservative reliability estimate because such an upper bound is used for the failure rate of the structure.

The mortality distributions $f_F(t)$ and $f_C(t)$ under failure mode F and C are respectively defined such that

$$f_F(t)dt = L(t)h_F(t)dt = \frac{\alpha}{V_F}\left(\frac{t}{V_F}\right)^{\alpha-1} \exp\left[-\left(\frac{t}{V_C}\right)-\left(\frac{t}{V_F}\right)^{\alpha}\right] \quad (9)$$

= probability that the structure will fail at O_F in $[t,t+dt)$

$$f_C(t)dt = L(t)h_C(t)dt = \frac{1}{V_C} \exp\left[-\left(\frac{t}{V_C}\right)-\left(\frac{t}{V_F}\right)^{\alpha}\right] \quad (10)$$

= probability that the structure will fail at O_C in $[t,t+dt)$

whereas the mortality distribution f(t) of the structure regardless of the mode of failure is

$$f(t) = f_F(t) + f_C(t) = L(t)h(t)$$

$$= \left[\frac{1}{V_C} + \frac{\alpha}{V_F}\left(\frac{t}{V_F}\right)^{\alpha-1}\right] \exp\left[-\left(\frac{t}{V_C}\right)-\left(\frac{t}{V_F}\right)^{\alpha}\right] \quad (11)$$

Define

$$P_F = \int_0^\infty f_F(t)dt \quad (12)$$

= probability that the structure will fail at O_F

$$P_C = \int_0^\infty f_C(t)dt \quad (13)$$

= probability that the structure will fail at O_C

Then, $P_F + P_C = \int_0^\infty f(t)dt$ and therefore

$$P_F + P_C = 1 \quad (14)$$

This indicates the condition that the failure, whether at O_C or at O_F will eventually occur.

Let T denote a random variable representing the life of the structure. Then, the (conditional) expectation of T of the structure which will fail due to fatigue is

$$E_F T = \int_0^\infty t\, f_F(t)dt \,/\, P_F \tag{15}$$

Similarly, the (conditional) expectation of T of the structure which will fail under mode C is

$$E_C T = \int_0^\infty t\, f_C(t)dt \,/\, P_C \tag{16}$$

The expected value of the life of the structure is

$$ET = \int_0^\infty t\, f(t)dt = P_F . E_F T + P_C . E_C T \tag{17}$$

By performing integration by parts in Eq. (17) for the particular form of f(t) given in Eq. (11), one can show that

$$ET = V_C P_C \tag{18}$$

Note that V_C is the expected life of a (hypothetical) fatigue-free "reference" structure and therefore P_C indicates the reduction in the expected life (in fraction) by adding a location (or a structural component) sensitive to fatigue failure to the reference structure. Probabilistically, this is a "series combination" of components since the structure is assumed to fail if one of these components fails.

To introduce the concept of the time to first failure, consider a group of n structures, each subjected to an independent sequence of load, and define

$$L_F^{(n)}(t) = \{L_F(t)\}^n \tag{19}$$

= probability that n chance-failure-free reference structures will all survive the interval [0,t) or equivalently probability that the first failure of these n reference structures will not occur in [0,t)

$$L_C^{(n)}(t) = \{L_C(t)\}^n \tag{20}$$

= probability that n fatigue-failure-free structures will all survive the interval [0,t) or equivalently probability that the first failure of these n reference structures will not occur in [0,t)

$$L^{(n)}(t) = L_F^{(n)}(t)L_C^{(n)}(t) = [L_F(t)L_C(t)]^n \tag{21}$$

= probability that the first failure, whether under mode F or C will not occur in [0,t)

It follows from Eqs. (19-21) that the risk functions $h_F^{(n)}(t)$ and $h_C^{(n)}(t)$ for the first failure at 0_F and 0_C respectively and $h^{(n)}(t)$ for the first failure regardless of the failure mode, are given as follows.

$$h_F^{(n)}(t) = nh_F(t) = -\frac{d\ \ell n\ L_F^{(n)}(t)}{dt} = \frac{n\alpha}{V_F}\left(\frac{t}{V_F}\right)^{\alpha-1} \tag{22}$$

$$h_C^{(n)}(t) = nh_C(t) = -\frac{d\ \ell n\ L_C^{(n)}(t)}{dt} = \frac{n}{V_C} \tag{23}$$

$$\begin{aligned} h^{(n)}(t) &= nh(t) = n[h_F(t) + h_C(t)] \\ &= -\frac{d\ \ell n\ L^{(n)}(t)}{dt} = \frac{n}{V_C} + \frac{n\alpha}{V_F}\left(\frac{t}{V_F}\right)^{\alpha-1} \end{aligned} \tag{24}$$

in which $h_F^{(n)}(t)dt(h_C^{(n)}(t)dt)$ is the conditional probability that the first failure will occur in [t,t+dt) under mode F(C) given that none of n chance-failure-free (fatigue-failure-free) reference structures has failed in the interval [0,t) whereas $h^{(n)}(t)dt$ is the conditional probability that the first failure will occur in the same interval [t,t+dt) whether under mode F or C given that none of the n structures has failed in the interval [0,t).

The mortality distributions $f_F^{(n)}(t)$ and $f_C^{(n)}(t)$ under mode F and C are defined such that

$$f_F^{(n)}(t)dt = L^{(n)}(t)h_F^{(n)}(t)dt = n\{L(t)\}^n h_F(t)dt \tag{25}$$

= probability that the first failure will occur at O_F in $[t,t+dt)$

$$f_C^{(n)}(t)dt = L^{(n)}(t)h_C^{(n)}(t)dt = n\{L(t)\}^n h_C(t)dt \tag{26}$$

= probability that the first failure will occur at O_C in $[t,t+dt)$

whereas the mortality distribution $f^{(n)}(t)$ of the first failure regardless of the mode is

$$f^{(n)}(t)dt = \{f_F^{(n)}(t) + f_C^{(n)}(t)\}dt = n\{L(t)\}^n h(t)dt \tag{27}$$

= probability that the first failure will occur either at O_C or at O_F in $[t,t+dt)$

Then,

$$P_F^{(n)} = \int_0^\infty f_F^{(n)}(t)dt = \frac{n\alpha}{V_F}\int_0^\infty \left(\frac{t}{V_F}\right)^{\alpha-1} e^{-n[(\frac{t}{V_C})+(\frac{t}{V_F})^\alpha]} dt \tag{28}$$

= probability that the first failure will occur under mode F

$$P_C^{(n)} = \int_0^\infty f_C^{(n)}(t)dt = \frac{n}{V_C}\int_0^\infty e^{-n[(\frac{t}{V_C})+(\frac{t}{V_F})^\alpha]} dt \tag{29}$$

= probability that the first failure will occur under mode C

As before,

$$P_F^{(n)} + P_C^{(n)} = \int_0^\infty [f_F^{(n)}(t) + f_C^{(n)}(t)]dt = 1 \tag{30}$$

indicating that the first failure will occur eventually.

Let $T^{(n)}$ denote a random variable representing the time to first failure among the group of n structures. Then, the (conditional) expectation of $T^{(n)}$ of the group in which the first

failure occurred under mode F is

$$E_F T^{(n)} = \int_0^\infty t\, f_F^{(n)} dt \,/\, P_F^{(n)} \tag{31}$$

Similarly, the (conditional) expectation of $T^{(n)}$ of the group in which the first failure occurred under mode C is

$$E_C T^{(n)} = \int_0^\infty t\, f_C^{(n)} dt \,/\, P_C^{(n)} \tag{32}$$

The expected value of the time to first failure is

$$ET^{(n)} = \int_0^\infty t\, f^{(n)}(t) dt = P_F^{(n)} \cdot E_F T^{(n)} + P_C^{(n)} \cdot E_C T^{(n)} \tag{33}$$

For the particular forms of risk functions assumed in Eqs. (1) and (2), it can be shown that

$$ET^{(n)} = \frac{V_C}{n} P_C^{(n)} = ET_C^{(n)} . P_C^{(n)} \tag{34}$$

where the well-known fact that V_C/n is the expected time to first failure in a group of n hypothetical fatigue-free "reference" structure, is used. Hence, $P_C^{(n)}$ indicates the reduction in terms of fraction in the expected time to first failure among the group by adding a location (or a structural component) sensitive to fatigue failure to each of these reference structures.

It is also pointed out that by setting $n = 1$, the expressions given in Eq. (19)-(34) produce the obvious identities $L_F(t) = L_F^{(1)}(t)$, $L_C(t) = L_C^{(1)}(t)$, $h_F(t) = h_F^{(1)}(t)$, $h_C(t) = h_C^{(1)}(t)$, etc.

In the following analysis, the probability $P_C^{(n)}$ plays an important role and therefore is evaluated here.

Setting $\xi = t/V_C$, $\gamma = V_F/V_C$ and $\lambda = \gamma n^{1-1/\alpha}$ in Eq. (29), the probability can be written as

$$P_C^{(n)} = \lambda \int_0^\infty e^{-\lambda \xi - \xi^\alpha} d\xi = \int_0^\infty e^{-\eta - (\eta/\lambda)^\alpha} d\eta \tag{35}$$

For $\alpha = 2$, the integrals in Eq. (35) can be carried out and

$$P_C^{(n)} = \sqrt{\pi}\,\lambda e^{\lambda^2/4}[1 - \Phi(-\frac{\lambda}{\sqrt{2}})] \tag{36}$$

where $\Phi(x)$ is the standardized normal distribution function.

For $\alpha = 2$, using Eq. (36) and for $\alpha = 3$, 4 and 5, evaluating the integral numerically on an IBM 7094, the value of $P_C^{(n)}$ is computed as a function of λ and plotted in Fig. 1. Also, the values of $P_C^{(n)}$ for various values of n, α and γ are computed and listed in Table I.

For small values of λ, $P_C^{(n)}$ can be approximated by taking first a few term of the following asymptotic expansion (take the second member of Eq. (35), expand $e^{-\lambda\xi}$ and integrate term by term),

$$P_C^{(n)} = \frac{1}{\alpha}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\lambda^k}{(k-1)!}\Gamma(\frac{k}{\alpha}) \qquad (1/\lambda \to \infty) \tag{37}$$

with $\Gamma(\cdot)$ being the gamma function, whereas for large values of λ, it has the following asymptotic expansion (take the third member of Eq. (35), expand $e^{-(\eta/\lambda)^\alpha}$ and integrate term by term),

$$P_C^{(n)} = 1 - \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!\,\lambda^{k\alpha}}\Gamma(1+k\alpha) \qquad (\lambda \to \infty) \tag{38}$$

Similar asymptotic expansions can be obtained for $E_C T^{(n)}$ and $E_F T^{(n)}$;

$$P_C^{(n)}\frac{E_C T^{(n)}}{ET_C^{(n)}} = \frac{1}{\alpha}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\lambda^{k+1}}{(k-1)!}\Gamma(\frac{k+1}{\alpha}) \qquad (1/\lambda \to \infty) \tag{39}$$

$$= 1 - \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!\,\lambda^{k\alpha}}\Gamma(2+k\alpha) \qquad (\lambda \to \infty) \tag{40}$$

$$P_F^{(n)}\frac{E_F T^{(n)}}{ET_F^{(n)}} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}\lambda^{k}}{(k-1)!}\Gamma(1+k/\alpha) \qquad (1/\lambda \to \infty) \tag{41}$$

$$= \alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k-1)!\ \lambda^{k\alpha}} \Gamma(1 + k\alpha) \qquad (k \to \infty) \qquad (42)$$

It can be shown that the compatible use (Eqs. (37), (39) and (41) together or Eqs. (38), (40) and (42) together) of these asymptotic expressions satisfy the identity given in Eq. (33).

In Appendix, the expression for the coefficient of variation of $T^{(n)}$ is derived and computed.

SECTION III

FATIGUE SENSITIVITY

The concept of the fatigue sensitive structure was first introduced by Freudenthal[3] who indicated a number of possible ways of defining the fatigue sensitivity factor. For example, take a structure which is subjected to two modes of failure as discussed in the preceding section. Consider the expected life ET_C of this structure disregarding the possible failure under the failure mode F (hypothetical fatigue-free reference structure);

$$ET_C = V_C \tag{43}$$

Similarly, consider the expected life ET_F of the same structure disregarding the possible failure under the mode C (hypothetical structure free of chance failure);

$$ET_F = V_F \Gamma(1 + 1/\alpha) \tag{44}$$

Then, it is possible to define ET_C/ET_F as a measure of fatigue sensitivity.[1]

However, a different definition of the fatigue sensitivity can be introduced through the following ratio k_C.

$$k_C = (ET_C - ET)/ET_C = 1 - ET/ET_C \qquad (0 \le k_C \le 1) \tag{45}$$

where ET is the expected life of the structure as defined in Eq. (17).

The significance of this ratio is rather evident; shorter the expected life of the actual structure subjected to both fatigue and chance failure compared with that of the reference (fatigue-free) structure, the larger k_C. Two limiting cases are of interest; if $ET = 0$, then $k_C = 1$ (completely sensitive) and if $ET = ET_C$, then $k_C = 0$ (completely insensitive).

Using Eqs. (18), (43) and (45),

$$k_C = 1 - P_C \tag{46}$$

Similarly, one can consider the ratio $k_C^{(n)}$ which defines the fatigue sensitivity of a group of n structure;

$$k_C^{(n)} = (ET_C^{(n)} - ET^{(n)})/ET_C^{(n)} \qquad (47)$$

where $ET_C^{(n)}$ is, as defined previously, the expected value of the time to first failure within the group of n reference (fatigue-free) structures.

Using Eqs. (34) and (47), and the fact that $ET_C^{(n)} = V_C/n$,

$$k_C^{(n)} = 1 - P_C^{(n)} \qquad (48)$$

Here again, an identity $k_C = k_C^{(1)}$ is obtained (since $P_C = P_C^{(1)}$) as it should when one set n = 1 in Eq. (47).

The effect of the group size n on the fatigue sensitivity $k_C^{(n)}$ is shown in Fig. 2 for γ = 1.0, 0.1 and 0.01 which indicates that with all other parameters being identical, the sensitivity decreases as any one of n, γ and α increases. However, when $\gamma \ll 1$ (as in the later example in Section IV), it seems more convenient to consider "chance-failure" sensitivity $k_F^{(n)}$ rather than "fatigue" sensitivity $k_C^{(n)}$ since, with $\gamma \ll 1$, the life of the structure is essentially controlled by fatigue failure process. For this purpose, define

$$k_F^{(n)} = (ET_F^{(n)} - ET^{(n)})/ET_F^{(n)} \qquad (49)$$

where $ET_F^{(n)}$ is the expected time to first failure among a group of n hypothetical "chance-failure-free" reference structures, and it can be shown[1] to be

$$ET_F^{(n)} = n^{-1/\alpha} ET_F = n^{-1/\alpha} V_F \Gamma(1 + 1/\alpha) \qquad (50)$$

Hence

$$k_F^{(n)} = 1 - P_C^{(n)}/\left(\lambda\Gamma(1 + 1/\alpha)\right) \qquad (51)$$

Equation (49) indicates that $k_F^{(n)} ET_F^{(n)}$ is the reduction in the expected time to first failure among a group of n chance-failure-free reference structures due to an addition (in series) of a

structural component sensitive to chance failure to each of these n structures. Therefore the closer $k_F^{(n)}$ to unity, the larger the effect of the chance failure component.

If Eq. (51) is considered only for small λ (this implies small γ), one can use the asymptotic expansion Eq. (37) for $P_C^{(n)}$ (take first two terms only) to produce

$$k_F^{(n)} = \lambda\Gamma(2/\alpha)/\Gamma(1/\alpha) \qquad (1/\lambda \to \infty) \qquad (52)$$

Using either Eq. (51) together with Table I or Eq. (52) depending on how small λ is, the value of $k_F^{(n)}$ is computed and plotted in Fig. 3 as a function of n for $\gamma = 0.01$ and 0.001.

The diagram indicates that, with all other parameters fixed, $k_F^{(n)}$ increases as any one of n, γ and α increases. Therefore, the decrease in the fatigue sensitivity $k_C^{(n)}$ with increasing n as observed above, gives a somewhat false sense of security, since by increasing n, $k_F^{(n)}$ increases. For example, consider the case of $\gamma = 0.01$ with $\alpha = 3$. For $n = 1$, $k_F^{(n)} < 10^{-2}$ and therefore, the effect of the existence of chance-failure component may be neglected at least for the estimation of expected time to failure. However, for n = 100 and 1,000, $k_F^{(n)}$ are approximately 0.10 (n = 100) and 0.36 (n = 1,000). This implies that the expected time to first failure of a group of 100 structures is 10% less than that of a group of 100 (chance-failure-free) reference structures, and 36% less for a group of 1,000 structures.

SECTION IV

DESIGN AND RELIABILITY

The relationship between the conventional design procedure and the reliability is studied in this section, considering an (idealized) aircraft wing with two critical locations or components. It is assumed that the wing root on the upper surface (O_C) is subjected to a possibility of failure due to instability (chance failure) and the wing root on the lower surface (O_F) is sensitive to fatigue failure.

(a) Gust Data

The following gust data in the flight altitude 0 ~ 10,000 ft. used in Ref. 4 are employed here for numerical purposes. The distribution function F(u) of the gust velocity u (ft./sec.) is governed by

$$F(u) = 1 - e^{-0.19u} \tag{53}$$

when the gust is observed in a thunderstorm turbulence whereas

$$F(u) = 1 - 0.969e^{-0.344u} - 0.031e^{-0.208u} \tag{54}$$

when the gust is observed in a non-thunderstorm turbulence associated with general operation.

Furthermore, it is observed that the frequency of gust occurrence is 14 gusts or 7 gust cycles per mile in a thunderstorm turbulence and 0.4 gusts or 0.2 gust cycles per mile in a non-thunderstorm turbulence (here it is assumed for simplicity that an up-gust is followed by a down-gust and vice-versa). Therefore, if $\nu_o(u)$ denotes the number of gust cycles per mile with velocity larger than u, then for a thunderstorm turbulence,

$$\nu_o(u) = 7e^{-0.19u} \tag{55}$$

and for a non-thunderstorm turbulence,

$$\nu_o(u) = 0.2 \times 0.969e^{-0.344u} + 0.2 \times 0.031e^{-0.208u} \tag{56}$$

Under further observation that a typical aircraft flies, on the average, 0.10% of the total flight distance within a thunderstorm turbulence while 18% in a non-thunderstorm turbulence, one can combine Eqs. (55) and (56) to derive the number of gust cycles per hour, $\nu(u)$, with velocity larger than u;

$$\nu(u) = 0.001V \times 7e^{-0.19u} + 0.18V \times 0.2 \times 0.969e^{-0.344u}$$

$$+ 0.18V \times 0.2 \times 0.031e^{-0.208u} \qquad (57)$$

where V is the flight speed of the aircraft.

Assuming V = 400 miles/hour in the present study,

$$\nu(u) = 2.80e^{-0.19u} + 0.432e^{-0.208u} + 14e^{-0.344u} \qquad (58)$$

Since $\nu(u)$ is the expected number (per hour of flight) of gust cycles with velocity larger than u and since it is a small number compared with unity, one can interpret this as the probability that such gust cycles will occur in one hour of flight.

The relative frequency $\nu^*(u)$ of gust cycles with velocity larger than u can be obtained as

$$\nu^*(u) = \nu(u)/\nu(0) = 0.162e^{-0.190u} + 0.025e^{-0.208u} + 0.813e^{-0.344u} \qquad (59)$$

The values of $\nu(u)$ and $\nu^*(u)$ are tabulated in Table II and $\nu^*(u)$ is plotted in Fig. 4 as a function of u.

(b) Ultimate Strength Design

With these data at hand, consider the design of a wing under the specified limit gust load 66 ft./sec. with a safety factor 1.5.

A major assumption here is that the ultimate failure will occur, due to upward gust, at the wing root on the upper surface in the form of instability rather than at the wing root on the lower surface. Such a situation can arise when the necessity of assuring a reasonable fatigue performance results in a design of the wing root on the lower

surface with an ultimate strength corresponding to a safety factor larger than 1.5.

Under this assumption, the chance failure of the wing is controlled by the possible instability on the upper surface.

Hence, the limit gust velocity 66 ft./sec. and the nominal ultimate strength UFLN (in terms of the acceleration due to gravity g) of the wing is related by

$$1.5(1.0 + 66\ K_n) = \text{UFLN} \qquad (60)$$

where K_n is the gust sensitivity factor, a function of flight velocity, lift coefficient gradient, effective wing area, and, the number 1.0 in the parentheses represents the effect of the level flight.

Suppose that the wing is designed so that $K_n = 0.035$ (a representative value indicated in Ref. 4) and UFLN = 5.0 satisfying Eq. (60).

It is important to note at this point of discussion that Jablecki's study[5] and a more recent study by Freudenthal and Wang[6] indicate possible unconservative discrepancies between the nominal ultimate strength and the actual ultimate strength. From these studies, it is observed that 50% of (structural) specimens sustained approximately 95% of the nominal strength or 0.95 UFLN is the median value of the statistical distribution of the actual ultimate strength. Therefore, taking 0.95UFLN = 4.75 as the actual ultimate strength UFL, the gust velocity u_D at which the ultimate failure will occur is computed;

$$1.0 + u_D K_n = 4.75 \quad \text{or} \quad u_D = 107 \text{ ft./sec.} \qquad (61)$$

From Fig. 4, one can evaluate the relative frequency ν_D^* of gust cycles with velocity larger than u_D = 107 ft./sec.; $\nu_D^* = 2.52 \times 10^{-10}$. Multiplying this value by $\nu(0)$ = 17.2/hour, the frequency of gust cycles with velocity larger than u_D or the probability of ultimate failure is obtained as

$$p = 4.33 \times 10^{-9}/\text{hour} \qquad (62)$$

and therefore

$$V_C = 1/(4.33 \times 10^{-9}) = 2.31 \times 10^{8} \text{ hours} \tag{63}$$

In passing, it is pointed out that this value of V_C agrees with $V_C = 7.14 \times 10^7$ hours for L-188 Electra and $V_C = 2.38 \times 10^8$ hours for L-749 Constellation, estimated independently by Lockheed.[7]

(c) Fatigue Design

In order to consider a numerical example of fatigue design, a typical S-N diagram for fatigue failure of Mustang wings with 20% of the ultimate load as the mean load is taken from Ref. 8 and replotted in Fig. 5. The wing under consideration is assumed to have this particular S-N diagram. The alternate load S about the mean load is also given in terms of percentage of the ultimate load and N is the number of cycles of S the wing can sustain before a fatigue failure. If one justifiably interprets the ultimate load used in Ref. 8 as the nominal ultimate load of the wing, the mean load equal to 20% of the ultimate load is consistent with the result of the preceding ultimate strength design in which the nominal ultimate load is 5(g) whereas the load due to the level flight is 1(g).

The statistical distribution of N at a constant load level S is assumed to be of the Weibull type.

$$F_N(x) = 1 - \exp[-\{(x-N_o)/(N_e-N_o)\}^{\alpha}] \tag{64}$$

where N_o, N_e and α are constant. In this connection, it is pointed out that the S-N diagram in Fig. 5 is interpreted as the S-EN diagram with EN representing the expected value of N, although this is not clearly indicated in Ref. 8. Moreover it is assumed, as in Ref. 1, on the basis of the experimental evidence[10] that (1) the distribution function of the fatigue life N under random load is also of the Weibull type, (2) the Palmgren-Miner rule or its modified version can be applied to estimate the expected value of the life under random fatigue using S-EN diagram and (3) for mathematical expedience, the scale parameter α of the distribution function is assumed to be a constant. For

the present study, $\alpha = 3$ is assumed. This is equivalent to the assumption that the standard deviation δ of the (common) logarithm of the life N is[11]

$$\delta = E\{\log_{10} N\} = \pi/(2.303\alpha\sqrt{3}) = 0.186 \qquad (65)$$

It is to be pointed out that, practically for all cases (whether it is constant load fatigue or random fatigue) the standard deviation δ is less than 0.186. It can be shown that this fact, combined with the further assumption that N_o in Eq. (64) is zero, will produce a conservative reliability estimate at least within the reliability range of interest. The assumption $N_o = 0$ reduces the form of the Weibull distribution Eq. (64) into that consistent with the failure rate given in Eq. (2) with conversion of the number of cycles (N) into the time (t), which in the present case takes the following form.

$$N = 17.2t, \; N_e = 17.2V_F \;\; (t, V_F \text{ in hours) etc.} \qquad (66)$$

Observing the S-N curve in Fig. 5, one can assume that the alternate load S less than 6% of UFL has no contribution to fatigue damage. Hence, constructing six (6) intervals of u beginning at u = 6 ft./sec., with interval length of 12 ft./sec. (except for the last) as shown in column (a) of Table III, the relative frequency p_i of the gust in each interval is computed using Fig. 4 and listed in column (d). The center of each of these intervals is given in terms of ft./sec. in column (b) and in terms of % UFL in column (c). In this conversion from (b) to (c), the fact is used that UFL = 5(g) and K_n = 0.035 (g/ft./sec.).

To estimate the expected fatigue life EN_R under random load, the Palmgren-Miner rule is employed. For this purpose, the expected fatigue life EN_i associated with the alternative load S_i indicated in column (c) is found by making use of the S-N diagram in Fig. 5 and listed in column (e).

According to the Palmgren-Miner rule,

$$\sum_{i=1}^{m} p_i EN_R/EN_i = k \qquad (67)$$

where m is the number of load levels or the intervals considered (m = 6 in this case) and k a parameter depending on material, structural configuration, etc.

With the aid of Table III

$$\sum_{i=1}^{5} p_i/EN_i = 2.11x10^{-7} + 2.58x10^{-7} + 7.39x10^{-8} + 1.60x10^{-8} + 5.06x10^{-9} = 5.64x10^{-7} \tag{68}$$

in which the effect of the last interval is neglected.

Assuming k = 1.0 in the present study (the standard Palmgren-Miner rule),

$$EN_R = 1.77x10^6 \tag{69}$$

which by virtue of Eq. (66), can be converted into the expected life ET_F in terms of hours

$$ET_F = EN_R/17.2 = 1.03x10^5 \text{ (hrs)} \tag{70}$$

Making use of the relation between ET_F and V_F given in Eq. (44) with $\alpha = 3.0$,

$$V_F = ET_F/\Gamma(1 + 1/\alpha) = 1.15x10^5 \text{ (hrs)} \tag{71}$$

Finally, it follows from Eqs. (63) and (71) that the parameter $\gamma = V_F/V_C$ defined in Section II is

$$\gamma = V_F/V_C = 1.15x10^5/(2.31x10^8) = 4.98x10^{-4} \tag{72}$$

At this point, it is emphasized that, although for design purposes, the use of the Palmgren-Miner rule and the like seems unavoidable, the validity of such a rule must be carefully examined by specimen and component fatigue test under random as well as constant loading conditions.

(d) Structural Reliability

Certain reliability statement can now be made on the basis of the preceding result.

First of all, the reliability $L^{(n)}(t)$ for a group of n such wings is given by Eq. (21). Explicitly, it is written as

$$L^{(n)}(t) = L_C^{(n)}(t) L_F^{(n)}(t) = \exp[-n(t/V_C) - n(t/V_F)^{\alpha}] \tag{73}$$

with $V_F = 1.15 \times 10^5$ hours, $V_C = 2.31 \times 10^8$ hours and $\alpha = 3.0$. This is plotted together with $L_C^{(n)}(t)$ and $L_F^{(n)}(t)$ in Fig. 6. It is interesting to note that the straight lines for $L_C^{(n)}(t)$ (chained) and $L_F^{(n)}(t)$ (dashed) are inscribed (from below) by the curves for $L^{(n)}(t)$ (solid).

Reading from Fig. 6, the time to first failure under the specified reliabilities 0.9, 0.99 and 0.999 are listed in Table IV.

With $\gamma = 4.98 \times 10^{-4}$ and $\alpha = 3$, $\lambda = \gamma n^{1-1/\alpha}$ and $P_C^{(n)}$ (using Eq. (37)) can be computed for various values of n (Table V).

Making use of Eq. (52), the chance failure sensitivity, as it is defined in Eq. (49), can be shown to be equal to λ multiplied by $\Gamma(2/3)/\Gamma(1/3) = 0.498$, which is small compared with unity. This implies that the expected time to first failure with n actual wings is practically the same as that with n chance-failure-free structures.

It should be emphasized, however, that this is true only when the expected values are compared. In fact, Fig. 6 clearly indicates that, if the group size n increases and/or higher reliability levels are desired, possibility of chance failure becomes a more dominant factor.

Since, obviously, the chance failure can no way be prevented by inspection, the wings have to be inspected for fatigue damage on the basis of the following (conditional) reliability $L_F^{*(n)}(t)$ involving the mortality distribution $f_F^{(n)}(t)$ in Eq. (25);

$$L_F^{*(n)}(t) = 1 - \frac{1}{P_F(n)} \int_0^t (n\alpha/V_F)(t/V_F)^{\alpha-1} e^{-n\{t/V_C + (t/V_F)^{\alpha}\}} dt \tag{74}$$

This provides the probability that the first failure "due to fatigue" will not occur before t. It can be shown that, for reliability

values close to unity (such as 0.99, 0.999 etc), the conditional reliability $L_F^{*(n)}(t)$ is practically identical with $L_F^{(n)}(t)$ (reliability of the first failure in a group of n chance-failure-free structures) indicated in Fig. 6 (dashed lines). Hence the times to first failure "due to fatigue" associated with reliabilities 0.9, 0.99 and 0.999 are read from Fig. 6 and listed in Table IV (in the parentheses). These values may be interpreted as first inspection-free period under the specified reliabilities. The difference between the figures without and within parentheses indicates the effect of chance failure. As pointed out above, it is larger, as n increases and/or higher reliability levels are demanded.

Other informations, particularly some of the expected values can also be obtained without difficulty. Making use of Eq. (34) and asymptotic expressions in Eqs. (39) and (41), $ET^{(n)}$, $E_CT^{(n)}$ and $E_FT^{(n)}$ can be evaluated for various values of n (Table V). This result indicates, for example, that the conditional expected time to first failure under chance mode is 5.77×10^3 hours for n = 1,000, a value much smaller than 1.045×10^4 hours, the conditional expected time to first failure due to fatigue. However, the probability of having chance failure, 4.45×10^{-2}, is small compared with that of fatigue failure, $1-4.45 \times 10^{-2}$.

SECTION V

DISCUSSION

(a) In the preceding sections, an idealized structure is considered in which only two locations have possibilities of failure. In reality, however, a major structure or structural component contains a number of critical locations. If such locations and their failure modes are identified using the engineering judgement based on the experience as well as on available theoretical and experimental knowledge (the aircraft industry is, in fact mostly responsible for this phase of study because of their experience and familiarity with the structural details), the failure rate of the structure may be estimated as the sum of the failure rates associated with individual failure locations. It can be shown from the addition rule of probability, that the failure rate thus obtained is a conservative estimate. However, it may be overly conservative. For example, if locations $0_1, 0_2, \ldots, 0_n$ are subjected to possible chance failures with probabilities of failure $p_1, p_2, \ldots, p_n$ per hour and if the chance failures considered are due to the same sequence of load, as in the preceding sections, these events are completely dependent and the maximum value of p_i serves as the rate of chance failure of the entire structure. This argument, however, does not seem to apply to the fatigue failure, since the experimental evidence indicates that the dependence between fatigue processes of two specimens is weak even under the same sequence of load. Therefore, for fatigue failure, it is highly recommended to add the rates of fatigue failure at possible locations within the structure to obtain a conservative estimate of the rate of failure.

A direct consequence of this discussion is as follows; (1) the rate of failure of a wing, in which the fatigue failure is possible on the upper surface (in addition to the lower surface) due to the load during taxiing, taking-off and landing, is given by $1/V_C + \alpha \times (t/V_F)^{\alpha-1}/V_F + \alpha^*(t/V_F^*)^{\alpha-1}/V_F^*$ where α^* and V_F^* are the scale pa-

rameter and the characteristic value of the fatigue process of the upper surface (assumed here is that the chance failure in compression on the lower surface is negligible) (2) the rate of two such wings of an aircraft is $1/V_C + 2\alpha(t/V_F)^{\alpha-1} / V_F + 2\alpha^*(t/V_F)^{\alpha^*-1}/V_F^*$.

(b) Also, in the preceding sections, the wing root on the lower surface is assumed to have the fatigue failure rate of the form $\alpha(t/V_F)^{\alpha-1}/V_F$ which is zero at $t = 0$. This is not a correct model since it could fail at $t = 0+$ under an extremely severe load. The probability $1/V_C^*$ per hour of such a failure, however, is assumed (Section IV, (b)) to be much smaller than $1/V_C$ of the upper surface and hence on the basis of the discussion (a) above, it is neglected in the analysis.

(c) In order to take into consideration the fact that the ultimate strength has a statistical distribution, 95% of nominal ultimate strength, a median value of its empirical distribution, is used in the analysis as deterministic strength (Section IV (b)). Evidently, this is an approximation. The method is well known to deal with such statistical variation of the resisting strength for a more rigorous reliability analysis (for example, Ref. 4). However, such a rigorous treatment is not performed here because (1) in view of the relatively larger statistical scatter in gust load distribution, the approximation seems reasonable and (2) it may make the analysis extremely lengthy and cumbersome, if not unmanageable, and hence obscure the essential purpose of the present study.

(d) The same reasoning is also valid for other simplifying assumptions made in the preceding analysis. For example, the following possibilities are tacitly disregarded; (1) the effect of the order of application of different load levels (this is always neglected if th Palmgren-Miner rule is used) and (2) the randomness in the number of gust cycles per hour. Also, (3) a quasi-static approach is employed here, although it is recognized that a more rigorous dynamic analysis treating a turbulent gust velocity as a stochastic

process (for example, Refs. 12 and 13) is possible and advisable to check the validity of the quasi-static analysis. However, it is pointed out that such a dynamic approach has its own difficulties. For example, usually the turbulent gust velocity and the response are assumed stationary and Gaussian. This is a questionable assumption, particularly when one is interested in a chance failure involving extreme values of the gust velocity. Furthermore, a chance failure implies the notorious first passage time problem when the load is treated as a stochastic process. Recent effort[14,15,16] to deal with these difficulties are noted here. It is hoped that the further study in this direction will make it possible to apply the dynamic analysis to realistic reliability problems

SECTION VI

CONCLUSIONS

On the basis of assumed additivity of the risks of failure, the distribution function of the time to first failure of a group of nominally identical structures, each with two critical locations respectively subjected to chance failure and fatigue failure, is derived with a careful discussion of the assumptions involved.

Various statistical quantities relevant to the time to first failure, such as (conditional) expected time to first failure due to chance failure or due to fatigue failure are defined and their analytical expressions are derived.

A definition of fatigue sensitivity, somewhat different from the one given by A. M. Freudenthal, is introduced together with the definition of the chance-failure sensitivity.

A realistic example involving aircraft wings designed according to the current method of design, indicates that when the group size n is increased or the desired level of reliability is raised, the chance failure becomes a dominant factor to be considered. Because of this, the usefulness of fatigue sensitivity or the chance-failure sensitivity is limited, if defined on the basis of expected values of the time to first failure. It is shown, however, that the (conditional) distribution of the time to first failure due to fatigue is almost identical with that associated with chance-failure-free structures for reliability values close to unity. Hence, the fatigue damage inspection may be planned on the basis of the distribution function of the time to first failure associated with these chance-failure-free structures.

Application of the present method to the estimation of the reliability or to the derivation of the distribution of the time to first failure under more realistic situations involving more than two critical failure locations, is also discussed.

APPENDIX

VARIANCE OF TIME TO FIRST FAILURE

The second moment of the time to first failure $T^{(n)}$ is given by

$$E\{T^{(n)}\}^2 = \int_0^\infty t^2 n\{1/V_C + \alpha/V_F(t/V_F)^{\alpha-1}\}e^{-n\{t/V_C+(t/V_F)^\alpha\}}dt \qquad \text{(A-1)}$$

which, after integration by parts and setting $\eta = nt/V_C$, becomes

$$E\{T^{(n)}\}^2 = 2(V_C/n)^2 \int_0^\infty \eta e^{-\eta-(\eta/\lambda)^\alpha} d\eta \qquad \text{(A-2)}$$

Since it can be shown that

$$P_C^{(n)} E_C T^{(n)} = (V_C/n) \int_0^\infty \eta e^{-\eta-(\eta/\lambda)^\alpha} d\eta \qquad \text{(A-3)}$$

Eq. (A-2) reduces to

$$E\{T^{(n)}\}^2 = 2(V_C/n) P_C^{(n)} E_C T^{(n)} = 2ET^{(n)} E_C T^{(n)} \qquad \text{(A-4)}$$

Therefore, the variance Var $T^{(n)}$ of $T^{(n)}$ is

$$\begin{aligned} \text{Var } T^{(n)} &= E\{T^{(n)}\}^2 - \{ET^{(n)}\}^2 \\ &= ET^{(n)}\{2E_C T^{(n)} - ET^{(n)}\} \end{aligned} \qquad \text{(A-5)}$$

and the coefficient of variation ν is

$$\nu = (2\, E_C T^{(n)}/ET^{(n)} - 1)^{1/2} \qquad \text{(A-6)}$$

Since $ET^{(n)} = P_C^{(n)} V_C/n$ has been computed in Section II, only $E_C T^{(n)}$ needs to be computed (from Eq. (A-3) on an IBM 7094). With $ET^{(n)}$ and $E_C T^{(n)}$ computed, v can be obtained from Eq. (A-6). Table VI lists the values of v.

Recall the fact[1] that the coefficients of variation of $T_C^{(n)}$ and $T_F^{(n)}$ are respectively unity and $[\Gamma(1+2/\alpha)/\Gamma^2(1+1/\alpha)-1]^{1/2}$, both

independent of n. The last expression is equal to 0.53, 0.36, 0.29 and 0.24 respectively for $\alpha = 2, 3, 4$ and 5.

Table VI, therefore, indicates that the coefficients of variation of $T^{(n)}$ are between those of $T_C^{(n)}$ and $T_F^{(n)}$.

REFERENCES

1. Freudenthal, A. M., "Reliability Analysis Based on Time to the First Failure", conference paper presented at the 5th I.C.A.F. Symposium, Melbourne, Australia, May 22-24, 1967.

2. Freudenthal, A. M., Itagaki, H., and Shinozuka, M., "Time to First Failure for Various Distributions of Time to Failure", Tech. Rep. No. AFML-TR-66-241, July 1966.

3. Freudenthal, A. M., "Fatigue Sensitivity and Reliability of Mechanical Systems, especially Aircraft Structures", WADD Tech. Rep. 611-53, July 1961.

4. Freudenthal, A. M., and Payne, A. O., "The Structural Reliability of Airframes", Tech. Rep. No. AFML-TR-64-401, December 1964.

5. Jablecki, L. S., "Analysis of the Premature Structural Failures in Static Tested Aircraft", Verlag Leemann, Zurich, 1955, p. 11 and p. 15.

6. Freudenthal, A. M., and Wang, P. Y., "Ultimate Strength of Aircraft Wing", Tech. Rep. No. AFML-TR-68-

7. Austin, W. H. Jr., "Development of Improved Gust Load Criteria for U.S. Air Force Aircraft," Annals of Reliability and Maintainability, Vol. 6, 1967, p. 70.

8. Payne, A. O., "Determination of the Fatigue Resistance of Aircraft Wings by Full-Scale Testing", in "Full Scale Fatigue Testing of Aircraft Structures", Pergamon Press, New York, 1960, p. 105.

9. Freudenthal, A. M.,"The Expected Time to First Failure", Tech. Rep. No. AFML-TR-66-37, Fig. 1, p. 14.

10. Freudenthal, A. M., and Heller, R. A., "On Stress Interaction in Fatigue and a Cumulative Damage Rule," Jour. of Aerospace Sciences, Vol. 26, No. 7, July 1959.

11. Gumbel, E. J., "Statistics of Extremes", Columbia University Press, New York, 1958, Eq. 4.2.4(7), p. 173.

12. Houbolt, J. C., Steiner, R., and Pratt, K. G., "Dynamic Response of Airplanes to Atmospheric Turbulence Including Flight Data on Input and Response", NASA TR R-199 (1964).

13. Steiner, R. and Pratt, K., "Some Applications of Power Spectra to Airplane Turbulence Problems", Jour. of Aircraft, Vol. 4, No. 4, July-August, 1967, pp. 360-365.

14. Lambert, R. F., Smits, T. I., and Langseth, R. E., "Some Non-Gaussian Noise Processes Relevent to Fatigue and Reliability Prediction", AFML TR 64-412, July 1965, AD 497-502.

15. Shinozuka, M., and Sato, Y., "Simulation of Nonstationary Random Process", Jour. of Engineering Mechanics Division, Proc. of ASCE, Vol. 93, No. EM1, February, 1967, pp. 11-40.

16. Shinozuka, M., Itagaki, H., and Hakuno, M., "Dynamic Safety Analysis of Multi-Story Buildings", Jour. of Structural Division, Proc. of ASCE, Vol. 94, No. ST1, January 1968.

TABLE I

Probability of Chance Failure $P_C^{(n)}$

(a) $\gamma = 0.01$

n \ α	2.0	3.0	4.0	5.0
1	0.00881	0.00888	0.00902	0.00914
10	0.0275	0.0405	0.0496	0.0562
100	0.0838	0.173	0.247	0.304
1000	0.237	0.569	0.779	0.885

(b) $\gamma = 0.1$

n \ α	2.0	3.0	4.0	5.0
1	0.0838	0.0849	0.0864	0.0875
10	0.237	0.332	0.393	0.435
100	0.545	0.816	0.921	0.963
1000	0.865	0.994	0.9997	0.999999

(c) $\gamma = 1.0$

n \ α	2.0	3.0	4.0	5.0
1	.546	.570	.584	.592
10	.865	.958	.984	.993
100	.981	.9994	.99998	-
1000	.998	.999994	-	-

TABLE II

Frequency $\nu(u)$ per hour of flight and relative frequency $\nu^*(u)$ of gust cycles with velocity larger than u.

u(ft./sec.)	$\nu(u)$ (cycles/hr.)	$\nu^*(u)$
0	17.2	1
20	8.42×10^{-2}	4.89×10^{-3}
40	1.52×10^{-3}	8.82×10^{-5}
60	3.33×10^{-5}	1.94×10^{-6}
80	7.31×10^{-7}	4.25×10^{-8}
100	1.62×10^{-8}	9.41×10^{-10}
120	3.59×10^{-10}	2.08×10^{-11}

(a) Interval u (ft/sec)	(b) Mid-Interval u (ft/sec)	(c) S % UFL	(d) Relative Frequency of Velocity in Interval	(e) Expected Fatigue Life (cycles)
(1) 6 ~ 18	12	$S_1 = 8.4$	$p_1 = 2.11 \times 10^{-1}$	$EN_1 = 10^6$
(2) 18 ~ 30	24	$S_2 = 16.8$	$p_2 = 1.16 \times 10^{-2}$	$EN_2 = 4.5 \times 10^4$
(3) 30 ~ 42	36	$S_3 = 25.2$	$p_3 = 7.39 \times 10^{-4}$	$EN_3 = 10^4$
(4) 42 ~ 54	48	$S_4 = 33.6$	$p_4 = 5.06 \times 10^{-5}$	$EN_4 = 3.16 \times 10^3$
(5) 54 ~ 66	60	$S_5 = 42$	$p_5 = 5.06 \times 10^{-6}$	$EN_5 = 10^3$
(6) 66 ~			$p_6 = 5.06 \times 10^{-7}$	$EN_6 < 10^3$

TABLE III

Gust Spectrum and Expected Fatigue Life

TABLE IV

Flying Hours Under Specified Reliability $L^{(n)}(t)$

$L^{(n)}(t)$ \ n	1	10	100	1,000
.9	5.38×10^4 (")	2.52×10^4 (")	1.17×10^4 (")	5.02×10^3 (5.38×10^3)
.99	2.40×10^4 (")	1.10×10^4 (")	4.80×10^3 (5.02×10^3)	1.59×10^3 (2.32×10^3)
.999	1.12×10^4 (")	5.02×10^3 (5.20×10^3)	1.59×10^3 (2.40×10^3)	2.12×10^2 (1.11×10^3)

TABLE V

Values of λ, $P_C^{(n)}$, $ET^{(n)}$, $E_CT^{(n)}$ and $E_FT^{(n)}$

n	λ	$P_C^{(n)}$	$ET^{(n)}$ (hr)	$E_CT^{(n)}$ (hr)	$E_FT^{(n)}$ (hr)
1	4.98×10^{-4}	4.45×10^{-4}	1.03×10^5	5.77×10^4	1.03×10^5
10	2.31×10^{-3}	2.06×10^{-3}	4.76×10^4	2.69×10^4	4.76×10^4
100	1.07×10^{-2}	9.55×10^{-3}	2.21×10^4	1.24×10^4	2.23×10^4
1000	4.98×10^{-2}	4.45×10^{-2}	1.03×10^3	5.77×10^3	1.045×10^4

TABLE VI

Coefficient of Variation v of

Time to First Failure $T^{(n)}$

(a) $\gamma = 0.01$

n \ α	2.0	3.0	4.0	5.0
1	0.530	0.360	0.290	0.240
10	0.532	0.382	0.310	0.268
100	0.550	0.444	0.418	0.417
1000	0.604	0.634	0.737	0.818

(b) $\gamma = 0.1$

n \ α	2.0	3.0	4.0	5.0
1	0.551	0.403	0.331	0.289
10	0.604	0.519	0.496	0.492
100	0.725	0.797	0.871	0.918
1000	0.893	0.989	0.999	0.999

(c) $\gamma = 1.0$

n \ α	2.0	3.0	4.0	5.0
1	0.725	0.632	0.596	0.579
10	0.893	0.931	0.959	0.974
100	0.982	0.996	0.997	0.997
1000	0.998	0.998	0.998	0.998

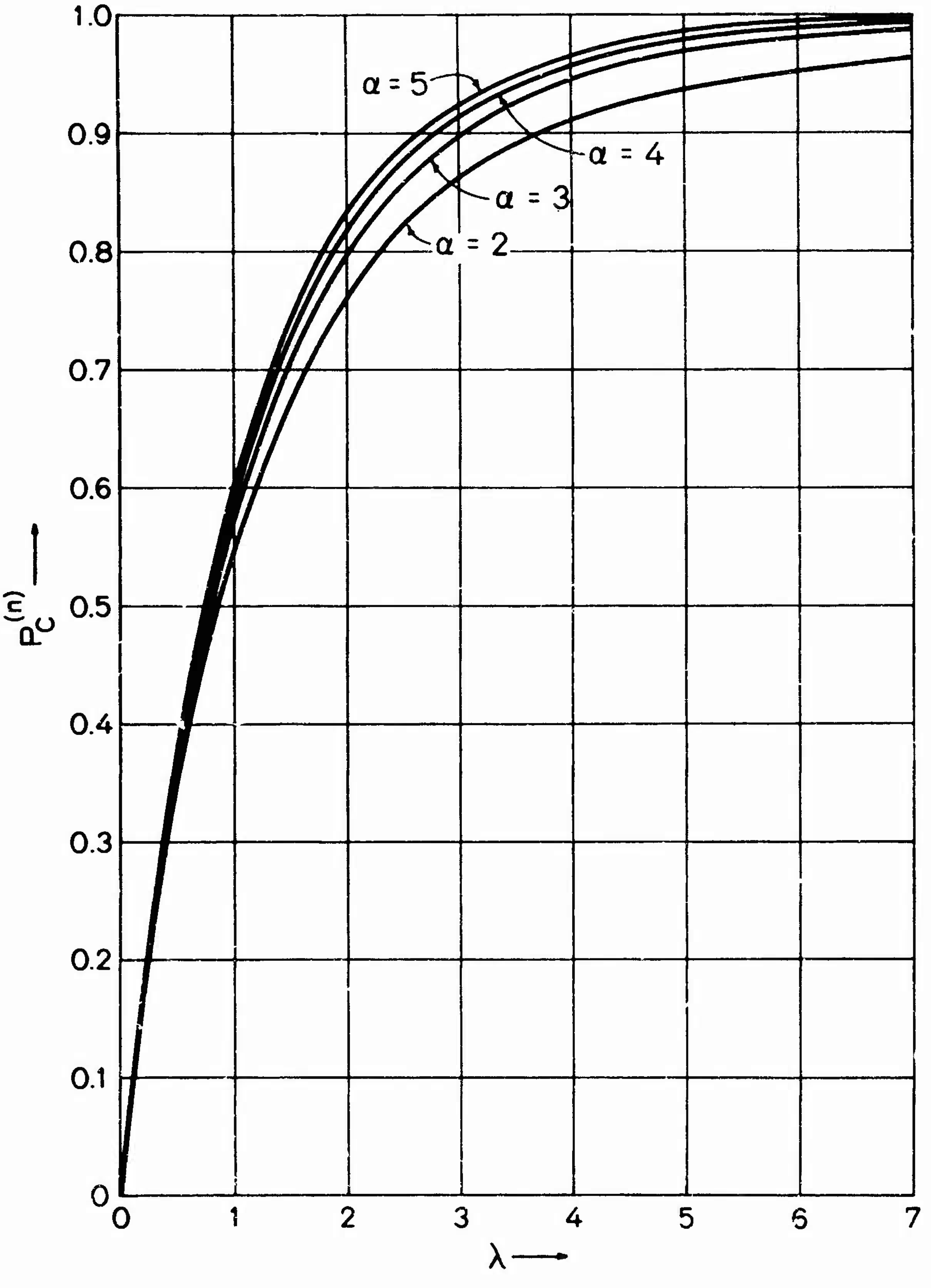

Fig. 1 Probability of chance failure $P_C^{(n)}$ as a function of λ.

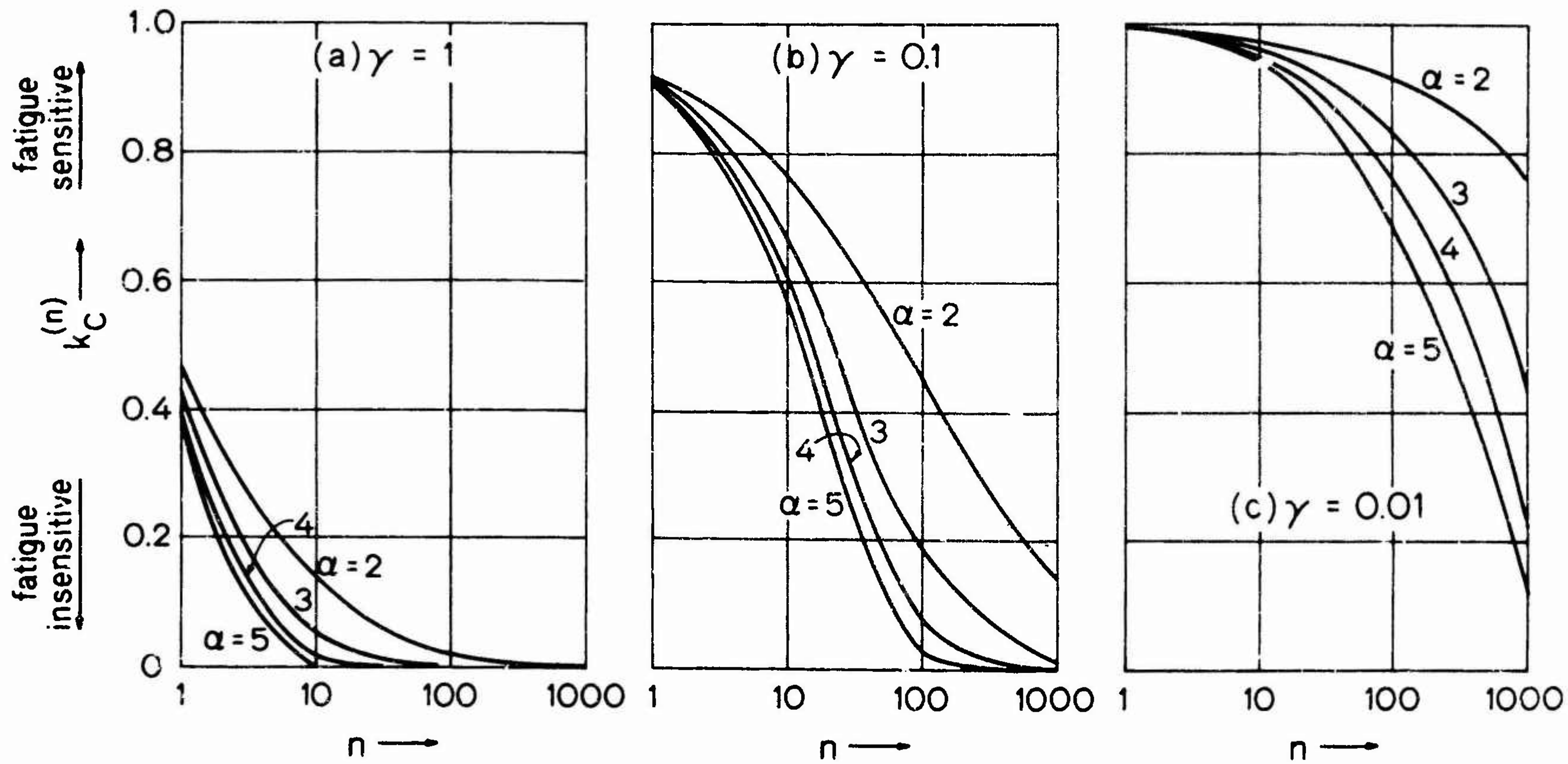

Fig. 2 Fatigue sensitivity $k_c^{(n)}$ as a function of group size n.

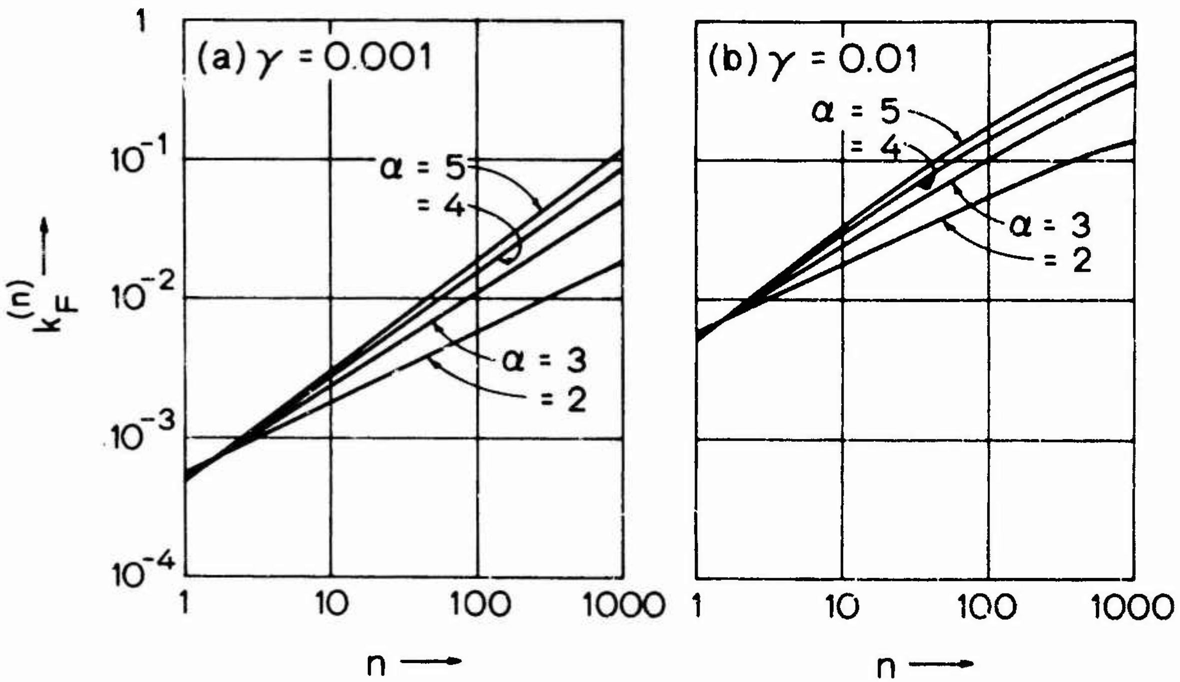

Fig. 3 Chance-failure sensitivity $k_F^{(n)}$ as a function of group size n.

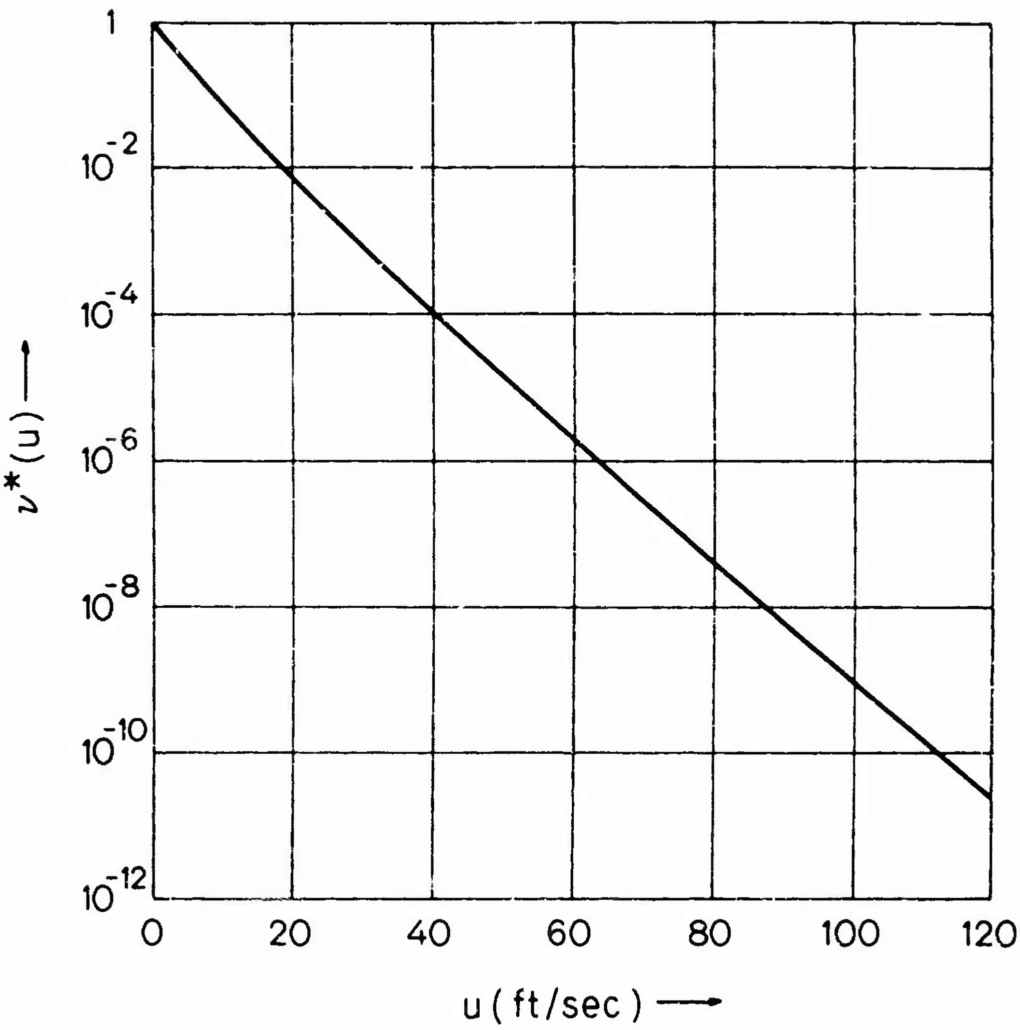

Fig. 4 Relative frequency $\nu^*(u)$ of gust cycles with velocity larger than u.

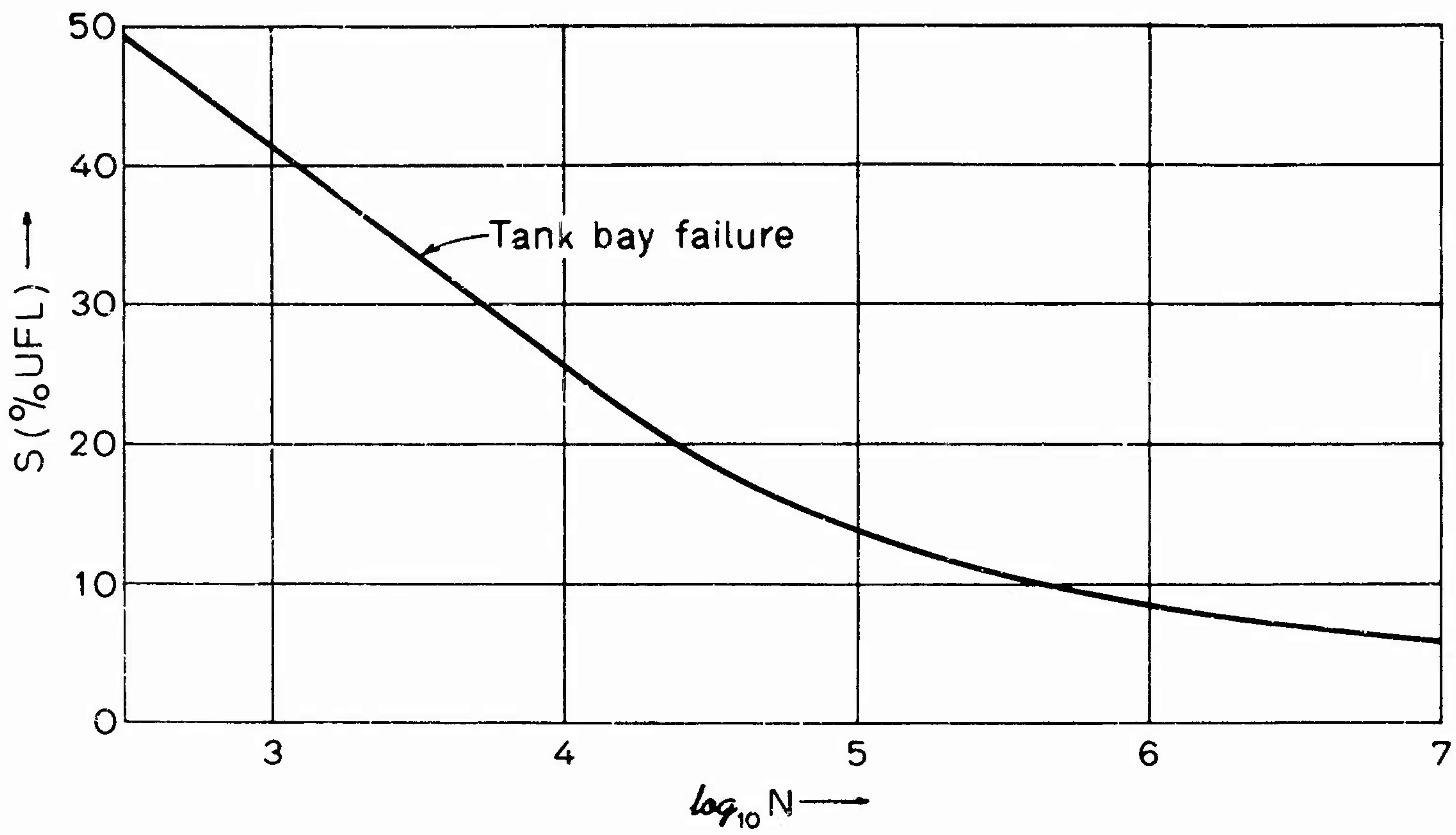

Fig. 5 S-N diagram for Mustang wings with mean load = 20% of ultimate failure load (UFL).

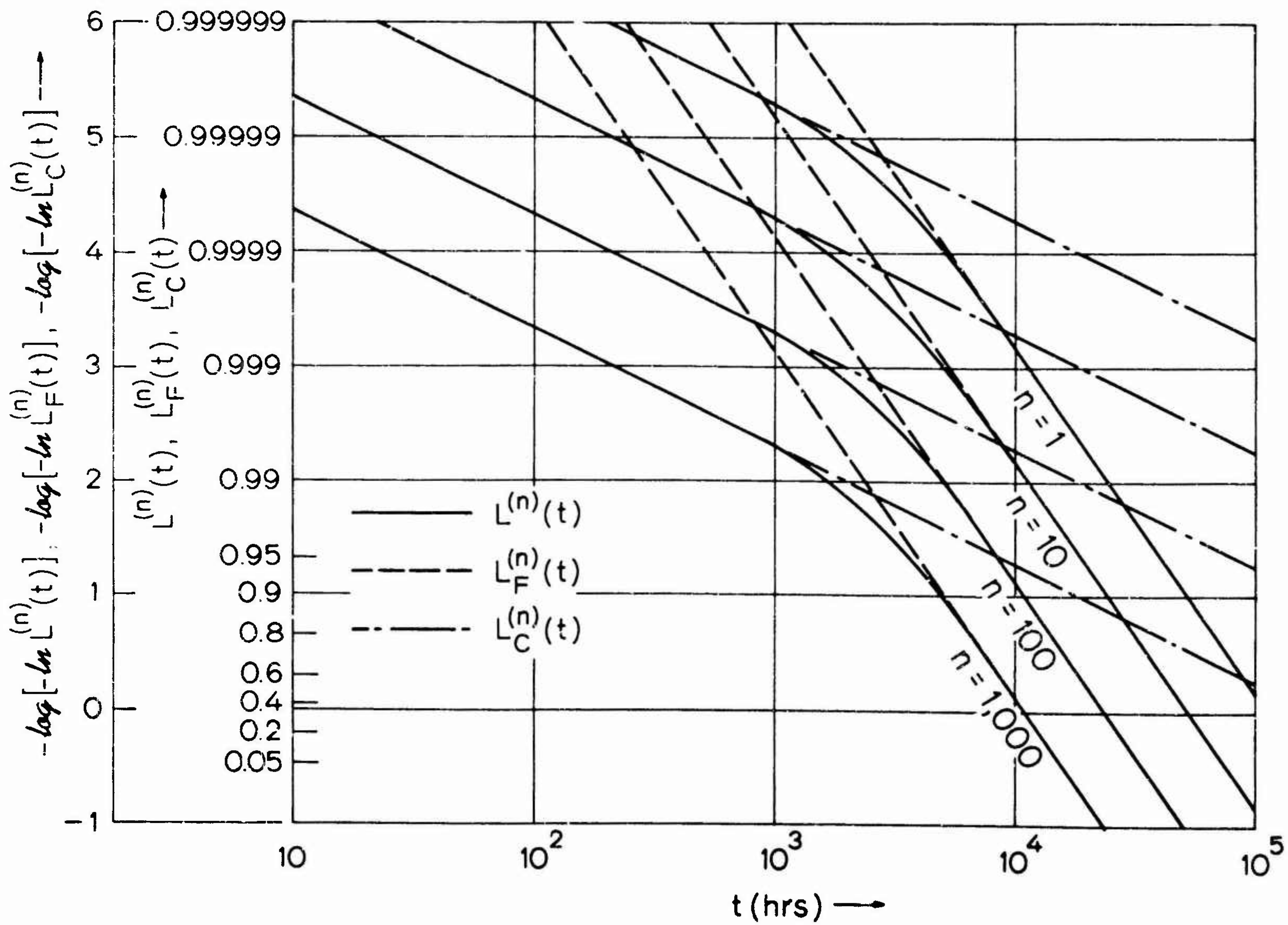

Fig. 6 Reliability $L^{(n)}(t)$, $L_F^{(n)}(t)$ and $L_C^{(n)}(t)$.

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13. ABSTRACT

On the basis of assumed additivity of risk functions, the statistical distribution of the time to first failure of a group of nominally identical structures, each with two critical locations respectively subjected to chance failure and fatigue failure, is derived together with analytical expressions of other statistical quantities pertinent to the problem of the time to first failure. The concept of fatigue sensitivity is examined. An example involving aircraft wings designed according to the current method of design is worked out to demonstrate how the reliability of such wings can be estimated. The example shows a dominance of chance failure if the group size increases and/or a higher reliability is demanded. It is pointed out that the present method of reliability estimation can easily be extended to cover more realistic structures.

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